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PERIODIC AND HOMOCLINIC SOLUTIONS OF THE MODIFIED $2+1$ CHIRAL MODEL

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Abstract. We use algebraic Bäcklund transformations (BTs) to construct explicit solutions of the modified $2+1$ chiral model from $T^2 \times \mathbb{R}$ to $SU(n)$, where T^2 is a 2-torus. Algebraic BTs are parameterized by $z \in \mathbb{C}$ (poles) and holomorphic maps π from T^2 to $\mathrm{Gr}(k, \mathbb{C}^n)$. We apply Bäcklund transformations with carefully chosen poles and π's to construct infinitely many solutions of the $2+1$ chiral model that are (i) doubly periodic in space variables and periodic in time, i.e., triply periodic, (ii) homoclinic in the sense that the solution u has the same stationary limit u_0 as $t \to \pm\infty$ and is tangent to a stable linear mode of u_0 as $t \to \infty$ and is tangent to an unstable mode of u_0 as $t \to -\infty$.

1. The $2+1$ Chiral model

A *wave map* $J : \mathbb{R}^{2,1} \to SU(n)$ is a critical point of the functional

$$\mathcal{E}(J) = \int_{\mathbb{R}^3} ||J^{-1}J_x||^2 + ||J^{-1}J_y||^2 - ||J^{-1}J_t||^2 \, dxdydt,$$

where $||\xi||^2 = -\mathrm{tr}(\xi^2)$, and x, y, t are the standard space-time variables. The Euler-Lagrange equation of $\mathcal{E}$ is

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y = 0. \tag{1.1}$$

This equation is also called *the* $2+1$ *chiral model*.

The *Ward equation* (or the *modified* $2+1$ *chiral model*) is the following equation for $J : \mathbb{R}^{2,1} \to SU(n)$:

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y - [J^{-1}J_t, J^{-1}J_y] = 0. \tag{1.2}$$

This equation is obtained by a dimension reduction and a gauge fixing of the self-dual Yang-Mills equation on $\mathbb{R}^{2,2}$ (cf. [11]). We call a solution of the Ward equation a *Ward map*. The Ward equation is completely integrable and many techniques from integrable systems can be used to construct explicit solutions.

We consider Ward maps satisfying the doubly periodic boundary condition in the space variables, i.e., Ward maps from $T^2 \times \mathbb{R}$ to $SU(n)$, where $T^2 = S^1 \times S^1$. Using the standard trick of writing a second order differential equation as a first order system on the tangent bundle of the phase space,

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we can view the Ward equation as a dynamical system on the tangent bundle $T(C^\infty(T^2, SU(n)))$. The goal of this paper is to construct periodic and homoclinic orbits of this dynamical system.

A Ward map $J : T^2 \times \mathbb{R} \to SU(n)$ independent of t is a harmonic map from T^2 to $SU(n)$. Equation for harmonic maps from T^2 to $SU(n)$ is integrable. Techniques from integrable systems were used to construct harmonic maps from T^2 to $SU(2)$ by Hitchin ([3]), and from T^2 to $SU(n)$ by Burstall, Ferus, Pedit and Pinkall ([1]).

A Ward map from $S^1 \times S^1 \times \mathbb{R}$ to $SU(n)$ independent of the second variable is a wave map from $S^1 \times \mathbb{R}$ to $SU(n)$. Such wave maps were studied by Terng and Uhlenbeck in [9].

A solution u of an evolution PDE is *homoclinic* if u tends to the same stationary solution u_0 as $t \to \pm\infty$ and is tangent to a stable linear mode of u_0 as $t \to +\infty$ and is tangent to an unstable linear mode of u_0 as $t \to -\infty$. The existence of homoclinic orbits for a finite dimensional dynamical system indicates the chaotic behavior of the system (cf. [4]). It is known that soliton equations in one space and one time variables (for example, sine-Gordon, KdV, and NLS), viewed as dynamical systems on certain function spaces, admit homoclinic orbits. Shatah and Strauss [6] proved that there are homoclinic wave maps from $S^1 \times \mathbb{R}$ to S^2, and Terng and Uhlenbeck [9] proved the same result for wave maps from $S^1 \times \mathbb{R}$ to any compact symmetric space. There have been many works concerning whether homoclinic orbits persist under small perturbation of these soliton equations in 1 space and 1 time variables (cf. [5, 7] and references therein).

One result of this paper is the existence of infinitely many Ward maps from $T^2 \times \mathbb{R}$ to $SU(n)$ that are periodic in time. In other words, we prove that there are infinitely many triply periodic solutions of the Ward equation. Another result of this paper is to show that the Ward equation has infinitely many homoclinic orbits. We give an outline of our method next.

The 1-soliton Ward maps from $\mathbb{R}^{2,1}$ to $SU(n)$ can be constructed as follows (cf. [11]). Let $z \in \mathbb{C}\backslash\mathbb{R}$ be a constant, $V = (v_{ij})$ a meromorphic map from $\mathbb{C}$ to the space $\mathcal{M}^0_{n\times k}$ of rank k complex $n \times k$ matrices, $\pi(x, y, t)$ the Hermitian projection of $\mathbb{C}^n$ onto the subspace spanned by the k columns of $V(w)$, where

$$w = x + \frac{(z - z^{-1})y}{2} + \frac{(z + z^{-1})t}{2}.$$

Let $\pi^\perp = \mathrm{I} - \pi$. Then

$$\hat{J}_{z,V}(x, y, t) = \pi^\perp(x, y, t) + \frac{\bar{z}}{z}\pi(x, y, t)$$

is a solution of the Ward equation. It has constant determinant, so we can normalized it to get a Ward map from $\mathbb{R}^{2,1}$ to $SU(n)$:

$$J_{z,V}(x, y, t) = \left(\frac{z}{\bar{z}}\right)^{k/n} \left(\pi^\perp(x, y, t) + \frac{\bar{z}}{z}\pi(x, y, t)\right).$$

$J_{z,V}$ (or $\hat{J}_{z,V}$) will be called a *Ward* 1-*soliton*. If all entries of $V(w)$ are rational functions in w, then $J_{z,V}$ is a smooth Ward map and is asymptotically constant as $|(x,y)| \to \infty$. If all entries of V are elliptic functions of same periods, then $J_{z,V}$ is a smooth Ward map from $T^2 \times \mathbb{R}$ to $SU(n)$.

Algebraic Bäcklund transformations (BTs) for the Ward equation were constructed in [2]. These are transformations that generate new Ward maps from a given Ward map and 1-solitons $J_{z,V}$ by a simple algebraic method.

We apply algebraic Bäcklund transformations repeatedly to 1-solitons associated to elliptic functions to construct infinitely many triply periodic Ward maps to $SU(n)$

Note that if the image of J lies in an abelian subgroup of $SU(n)$, then the Ward equation for J becomes the linear wave equation. For example, let m be an integer, and $a = \mathrm{diag}(im, -im)$. Then

$$J_0(x,y,t) = \exp(-(x+y)a)$$

is a doubly periodic, stationary Ward map, whose image lies in $SO(2)$.

We apply algebraic BTs $2k$ times to J_0 with carefully chosen poles and projections to construct homoclinic Ward maps J_{2k} from $T^2 \times \mathbb{R}$ to $SU(n)$, and prove that J_{2k} tends to $(-1)^k J_0$ as $|t| \to \infty$ and J_{2k} are homoclinic.

This paper is organized as follows: We review the Lax pair and algebraic Bäcklund transformations for the Ward equation in section 2, and use elliptic functions to construct triply periodic Ward maps in section 3. In the last section, we construct (i) homoclinic Ward maps from $T^2 \times \mathbb{R}$ to $SU(n)$ that tend to stationary solutions, (ii) homoclinic Ward maps that tend to periodic solutions.

2. Extended Ward maps and Bäcklund transformations

The Ward equation is integrable in the sense that it can be written as the compatibility condition for a system of linear equations involving a spectral parameter $\lambda \in \mathbb{C}$. In fact, we have the following theorem (cf. [11])

Theorem 2.1. *Let $J : \mathbb{R}^{2,1} \to SU(n)$ be a Ward map, $dx^2 + dy^2 - dt^2$ be the Lorentzian metric on $\mathbb{R}^{2,1}$,*

$$u = \frac{t+y}{2}, \quad v = \frac{t-y}{2}, \tag{2.1}$$

$A = J^{-1}J_u$, and $B = J^{-1}J_x$. Then the following linear PDE system is solvable for $\psi : \mathbb{R}^{2,1} \times \mathbb{C} \to GL(n,\mathbb{C})$:

$$\begin{cases} (\lambda\partial_x - \partial_u)\psi = A\psi, \\ (\lambda\partial_v - \partial_x)\psi = B\psi. \end{cases} \tag{2.2}$$

Conversely, suppose $\mathcal{O}$ is an open subset of 0 in $\mathbb{C}$ and $\psi : \mathbb{R}^{2,1} \times \mathcal{O} \to GL(n,\mathbb{C})$ is a smooth map so that

$$A := (\lambda\psi_x - \psi_u)\psi^{-1}, \quad B := (\lambda\psi_v - \psi_x)\psi^{-1}$$

are independent of $\lambda \in \mathcal{O}$ and ψ satisfies the $U(n)$-reality condition

$$\psi(x,u,v,\bar{\lambda})^*\psi(x,u,v,\lambda) = \mathrm{I}, \tag{2.3}$$

Then

$$J(x,y,t) = \psi(x,y,t,0)^{-1}$$

is a smooth solution of the Ward equation and $J^{-1}J_u = A$ and $J^{-1}J_x = B$.

A solution $\psi(x,y,t,\lambda)$ of (2.2) that satisfies the $U(n)$-reality condition (2.3) is called an *extended Ward map* and $J = \psi(\cdots,0)^{-1}$ the associated Ward map.

Given $z \in \mathbb{C}$ and a Hermitian projection π of $\mathbb{C}^n$, let

$$h_{z,\pi}(\lambda) = \pi^\perp + \frac{\lambda - z}{\lambda - \bar{z}}\pi = \mathrm{I} + \frac{\bar{z} - z}{\lambda - \bar{z}}\pi,$$

where $\pi^\perp = \mathrm{I} - \pi$. A direct computation implies that $h_{z,\pi}$ satisfies the $U(n)$-reality condition (2.3).

Let $V = (v_{ij}) : \mathbb{C} \to \mathcal{M}^0_{n\times k}(\mathbb{C})$ be a meromorphic map, and $\pi(x,y,t)$ the Hermitian projection onto the subspace spanned by the columns of $V(w)$, where

$$w = x + zu + z^{-1}v,$$

and u, v are the light cone coordinates in the yt-plane defined by (2.1). Since the entries of V are meromorphic functions, the projection π is smooth on $\mathbb{R}^{2,1}$. Set

$$\psi(x,y,t,\lambda) = h_{z,\pi(x,y,t)}(\lambda) = \pi^\perp(x,y,t) + \frac{\lambda - z}{\lambda - \bar{z}}\,\pi(x,y,t)$$

A direct computation implies that both $(\lambda\psi_x - \psi_u)\psi^{-1}$ and $(\lambda\psi_v - \psi_x)\psi^{-1}$ are independent of λ. By Theorem 2.1, ψ is an extended solution of the Ward equation and the associated Ward map is the 1-soliton

$$J_{z,V}(x,y,t) = \left(\frac{z}{\bar{z}}\right)^{k/n}\psi(x,y,t,0)^{-1} = \left(\frac{z}{\bar{z}}\right)^{k/n}\left(\pi^\perp(x,y,t) + \frac{\bar{z}}{z}\pi(x,y,t)\right),$$

where $\left(\frac{z}{\bar{z}}\right)^{k/n}$ is a normalizing constant to make $\det(J_{z,V}) = 1$.

The 1-soliton Ward map $J_{z,V}$ is a travelling wave because

$$w = x + zu + z^{-1}v = (x - v_1t) + k_1(y - v_2t) + ik_2(y - v_2t),$$

where $z = re^{i\theta}$, $v_1 = -\frac{2r\cos\theta}{1+r^2}$, $v_2 = \frac{1-r^2}{1+r^2}$, and $k_1 + ik_2 = (z - z^{-1})/2$. Thus $J_{z,V}$ is a travelling wave with constant velocity $\vec{v} = (-\frac{2r\cos\theta}{1+r^2}, \frac{1-r^2}{1+r^2})$ on the xy-plane. In particular, $J_{i,V}$ is a stationary Ward map, i.e., a harmonic map from $\mathbb{C}$ to $SU(n)$.

The following theorem was proved in [2], which gives an algebraic method to produce new extended Ward maps from a given one.

Theorem 2.2 (Bäcklund transformation)**.** *Let $\psi(x,y,t,\lambda)$ be an extended solution of the Ward equation and $J = \psi(\cdots,0)^{-1}$ the associated Ward map from $\mathbb{R}^{2,1}$ to $SU(n)$. Choose $z \in \mathbb{C}\setminus\mathbb{R}$ such that $\psi(x,y,t,\lambda)$ is holomorphic*

and non-degenerate at $\lambda = z$. *Let* $h_{z,\pi(x,y,t)}(\lambda)$ *be an extended* 1*-soliton solution, and* $\tilde{\pi}(x,y,t)$ *the Hermitian projection of* $\mathbb{C}^n$ *onto*

$$\psi(x,y,t,z)\mathrm{Im}(\pi(x,y,t)).$$

Then

$$\psi_1(x,y,t,\lambda) = h_{z,\tilde{\pi}(x,y,t)}(\lambda)\psi(x,,y,t,\lambda)$$

is a new extended solution to the linear system (2.2) with

$$(A,B) \to (A + (\bar{z}-z)\tilde{\pi}_x, B + (\bar{z}-z)\tilde{\pi}_v),$$

and the new Ward map is

$$J_1(x,y,t) = \left(\frac{z}{\bar{z}}\right)^{k/n} J(x,y,t)\ \left(\frac{\bar{z}}{z}\tilde{\pi}(x,y,t) + \tilde{\pi}^{\perp}(x,y,t)\right).$$

We will denote $\psi_1 = h_{z,\pi} * \psi$ and $J_1 = h_{z,\pi} * J$, the Bäcklund transformation generated by $h_{z,\pi}$.

3. Periodic Ward maps from $T^2 \times \mathbb{R}$ to $SU(n)$

We use algebraic BTs to construct Ward maps into $SU(n)$ that are either doubly periodic in space variables or triply periodic.

First we construct 1-soliton Ward maps that are doubly periodic. Let $z = re^{i\theta}$,

$$w(x,y,t) = x + zu + z^{-1}v = x + \frac{z - z^{-1}}{2}\ y + \frac{z+z^{-1}}{2}\ t,$$

and $\alpha = a + ib$. A direct computation shows that

$$w(x + a - \frac{k_1}{k_2}\ b,\ y + \frac{b}{k_2},\ t) = w(x,y,t) + \alpha,$$

where $k_1 + ik_2 = \frac{z-z^{-1}}{2}$. Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function such that $f(w+\alpha) = f(w)$ (i.e., periodic with period α), and

$$g(x,y,t) = f(w) = f(x + zu + z^{-1}v).$$

A direct computation shows that

$$g(x,y,t) = g(x + a - \frac{k_1}{k_2}b, y + \frac{b}{k_2}, t).$$

Hence

(1) if $\alpha = 2\pi$, then g is 2π-periodic in x;
(2) if $\alpha = 2\pi(k_1 + ik_2) = \pi(z - z^{-1})$, then g is 2π-periodic in y.

This shows that if each entry v_{ij} of the meromorphic map $V = (v_{ij}) : \mathbb{C} \to \mathcal{M}^0_{n\times k}(\mathbb{C})$ satisfies $v_{ij}(w + 2\pi) = v_{ij}(w + \pi(z - z^{-1})) = v_{ij}(w)$, i.e. an elliptic function with periods 2π and $\pi(z - z^{-1})$, then the 1-soliton $J_{z,V}$

is a doubly periodic Ward map with respect to the lattice $2\pi(\mathbb{Z}\times\mathbb{Z})$. An example of elliptic function is the well-known Weierstrass $\wp$-function

$$\wp(w) = \frac{1}{w^2} + \sum_{\gamma\in\Lambda\backslash\{0\}} \left(\frac{1}{(w-\gamma)^2} - \frac{1}{\gamma^2}\right),$$

where Λ is the lattice in $\mathbb{C}$ generated by 2π and $\pi(z-z^{-1})$. Other elliptic functions can be generated by Weierstrass $\wp$-functions and Jacobi elliptic functions. It is clear that $J_{z,V}$ is time periodic if and only if the ratio of the velocity, $v_1/v_2 = (-2r\cos\theta)/(1-r^2)$, is rational.

Similar computation implies that given any rank 2 lattice Λ of $\mathbb{C}$ there are 1-soliton Ward maps from $\mathbb{C}/\Lambda\times\mathbb{R}$ to $SU(n)$. Moreover, some of these 1-solitons are periodic in time, i.e., triply periodic. In particular, we get

Theorem 3.1. *Let $\tau = c_1 + ic_2$ with $c_2 \neq 0$, $\Lambda = \mathbb{Z}2\pi + \mathbb{Z}\tau$, $z = re^{i\theta}$ a constant, and $a + ib = c_1 + \frac{z-z^{-1}}{2}c_2$. If each entry v_{ij} of the meromorphic map $V:\mathbb{C}\to\mathcal{M}^0_{n\times k}(\mathbb{C})$ is an elliptic function with periods 2π and $a+ib$, then the extended 1-soliton solution $h_{z,\pi}$ is doubly periodic with periods 2π and τ and the associated 1-soliton*

$$J_{z,V} = e^{i2k\theta/n}h_{z,\pi}(0)^{-1} = e^{i2k\theta/n}(\pi^\perp + e^{-2i\theta}\pi)$$

is a Ward map from $\mathbb{C}/\Lambda\times\mathbb{R}$ to $SU(n)$, where $e^{i2k\theta/n}$ is a normalizing constant, and $\pi(x,y,t)$ is the projection onto the subspace spanned by the columns of $V(x+zu+z^{-1}v)$. Moreover,

(1) *if $r\neq 1$ and there exist integers m_1, m_2 such that*

$$\frac{2\cos\theta}{r-r^{-1}} = \frac{2\pi m_1 + m_2c_1}{m_2c_2},$$

then $J_{z,V}$ is periodic in time with period $T = \frac{m_2c_2(r+r^{-1})}{r-r^{-1}}$,

(2) *if $r = 1$ and $\cos\theta\neq 0$, then $J_{z,V}$ is periodic in time with period $T = \frac{2\pi}{\cos\theta}$.*

In the rest of the section we consider only the square torus. We will construct k-soliton Ward maps from $T^2\times\mathbb{R}$ to $SU(n)$ that are also time periodic. To do this, we define

$$\mathcal{Z} = \{z = re^{i\theta}\in\mathbb{C}\setminus\mathbb{R} \mid z = e^{i\theta}\neq\pm i \text{ or } \cos\theta/(r-r^{-1})\in\mathbb{Q}\},$$

where $\mathbb{Q}$ denotes the set of rational numbers. We have seen that for each $z\in\mathcal{Z}$, we can construct time periodic 1-solitons to the Ward equation. Moreover, the time period T depends on z only. In fact, the period function $T:\mathcal{Z}\to\mathbb{R}$ is defined as follows:

$$T(z) = \begin{cases} \frac{2\pi}{\cos\theta}, & \text{if } z = e^{i\theta}\neq\pm i, \\ \frac{2\pi m_2(r+r^{-1})}{r-r^{-1}}, & \text{if } z = re^{i\theta},\ \frac{2\cos\theta}{r-r^{-1}} = \frac{m_1}{m_2}. \end{cases} \tag{3.1}$$

Apply Bäcklund transformations repeatedly with some rational conditions on the poles $z_1,\ldots,z_m$ to get the following:

Theorem 3.2. *Let $\{z_1, \ldots, z_m\}$ be a set of finite points in $\mathcal{Z}$ such that $z_i \neq z_j, \bar{z}_j$ for all $i \neq j$, and $h_{z_i,\pi_i}(\lambda)$ extended 1-soliton solutions leading to doubly periodic Ward maps, where $i, j = 1, \cdots m$. Let $T_i = T(z_i)$ be the time period defined in (3.1). Let J_1 be the Ward map associated to h_{z_1,π_1}, i.e., $J_1 = h_{z_1,\pi_1}(0)^{-1}$. Let J_m be the Ward map obtained by applying $m-1$ Bäcklund transformations to J_1,*

$$J_m = h_{z_m,\pi_m} * (\cdots * (h_{z_2,\pi_2} * J_1)\cdots). \tag{3.2}$$

If T_j/T_1 are rational numbers for all $2 \leq j \leq m$, then J_m is a Ward map from $T^2 \times \mathbb{R}$ to $SU(n)$ and is periodic in time. In other words, J_m is a triply periodic solution of the Ward equation.

Proof. We prove the two-soliton case. By Theorem 2.2, we have

$$h_{z_2,\pi_2} * h_{z_1,\pi_1} = h_{z_2,\tilde{\pi}_2} h_{z_1,\pi_1},$$

where $\mathrm{Im}\tilde{\pi}_2 = h_{z_1,\pi_1}(z_2)\mathrm{Im}\pi_2 = (\mathrm{I} + \frac{\bar{z}_1 - z_1}{z_2 - \bar{z}_1}\pi_1)\mathrm{Im}\pi_2$. Note that $\tilde{\pi}_2$ is periodic in time because π_1 and π_2 are time periodic and T_2/T_1 is rational. Thus we see that $h_{z_2,\pi_2} * h_{z_1,\pi_1}$ is time periodic, and so is the associated Ward map. The general case can be proved by induction. □

4. Homoclinic Ward maps

It is known that solutions of the sine-Gordon equation (SGE)

$$q_{tt} - q_{xx} = \sin q$$

give rise to wave maps from $\mathbb{R}^{1,1}$ to S^2. Breather solutions are 2-soliton solutions of the SGE that are periodic in the x variable. Shatah and Strauss proved in [6] that wave maps from $S^1 \times \mathbb{R}$ to S^2 corresponding to breather solutions of the sine-Gordon equation are homoclinic wave maps. Applying Bäcklund transformation $2k$-times with carefully placed poles, Terng and Uhlenbeck constructed $2k$-soliton solutions for the sine-Gordon equation that are periodic in the space variable, and showed that the corresponding wave maps from $S^1 \times \mathbb{R}$ to S^2 are also homoclinic. More generally they proved that there are homoclinic wave maps from $S^1 \times \mathbb{R}$ into any compact symmetric space [9].

In this section, we apply Bäcklund transformations with carefully chosen poles and Hermitian projections even times to certain stationary wave map into $SO(2)$ to construct homoclinic Ward maps from $T^2 \times \mathbb{R}$ to $SU(n)$. To make the construction more illuminating, we will work on the $SU(2)$ model. The $SU(n)$ model is similar.

Let $m > 0$ be an integer, and $a = \mathrm{diag}(im, -im) \in su(2)$. It is easy to check that

$$\psi(\lambda)(x, y, t) = \psi(x, y, t, \lambda) = e^{((1-\lambda)x + (1+\lambda-\lambda^2)u - v)a}. \tag{4.1}$$

is an extended Ward map. So

$$J_0(x, y, t) = \psi(x, y, t, 0)^{-1} = e^{-(x+u-v)a} = e^{-(x+y)a}$$

is a stationary Ward map, which is doubly periodic in the space variables. Note that J_0 is a harmonic map from T^2 to $SO(2)$.

Next we compute the linearization of the Ward equation at the stationary solution $J_0 = e^{-(x+y)a}$, as well as its stable and unstable subspaces. Let $\mathcal{M} = C^\infty(T^2 \times \mathbb{R}, SU(2))$. Then we can give a natural trivialization of the tangent bundle $T\mathcal{M}$ as follows. Given a curve $\gamma : (-\epsilon, \epsilon) \to \mathcal{M}$ with $\gamma(0) = J$, we identify the tangent vector $\gamma'(0)$ as

$$(\gamma(0), \gamma(0)^{-1}\gamma'(0)) = (J, J^{-1}\delta J).$$

This identifies $T\mathcal{M} = \mathcal{M} \times C^\infty(T^2 \times \mathbb{R}, su(2))$.

Set $J^{-1}\delta J = \eta$. Compute directly to get

$$\begin{aligned}
\delta(J^{-1}J_x) &= -(J^{-1}\delta J)J^{-1}J_x + J^{-1}(\delta J)_x \\
&= -\eta(J^{-1}J_x) + J^{-1}(J\eta)_x = -\eta(J^{-1}J_x) + J^{-1}(J_x\eta + J\eta_x) \\
&= \eta_x + [J^{-1}J_x, \eta].
\end{aligned}$$

The computation for $\delta(J^{-1}J_y)$ and $\delta(J^{-1}J_t)$ is similar. So the linearization of the Ward equation at $J_0 = e^{-(x+y)a}$ is:

$$\begin{aligned}
& (\eta_t + [J^{-1}J_t, \eta])_t - (\eta_x + [J^{-1}J_x, \eta])_x - (\eta_y + [J^{-1}J_y, \eta])_y \\
& \quad -[\eta_t + [J^{-1}J_t, \eta], J^{-1}J_y] - [J^{-1}J_t, \eta_y + [J^{-1}J_y, \eta]] \\
= \; & \eta_{tt} - \eta_{xx} - \eta_{yy} + [a, \eta_x + \eta_y - \eta_t] = 0. \qquad (4.2)
\end{aligned}$$

We note that the linearization at $J = -e^{-(x+y)a}$ is the same one. Write (4.2) in terms of entries $\eta = \begin{pmatrix} ir & \xi \\ -\bar{\xi} & -ir \end{pmatrix}$ to get

$$\begin{cases} r_{tt} - r_{xx} - r_{yy} = 0, \\ \xi_{tt} - \xi_{xx} - \xi_{yy} + 2im(\xi_x + \xi_y - \xi_t) = 0. \end{cases} \qquad (4.3)$$

This system is linear with constant coefficients, so it can be solved by Fourier series. Let

$$\xi = \sum_{j,l \in \mathbb{Z}} b_{jl}(t) e^{i(jx+ly)}$$

be the Fourier series expansion of ξ. Then by (4.3.2), we have

$$b''_{jl} - 2imb'_{jl} + (j^2 + l^2 - 2m(j+l))b_{jl} = 0,$$

where $'$ means differentiation with respect to t. Its auxiliary equation is

$$\gamma^2 - 2im\gamma + j^2 + l^2 - 2m(j+l) = 0.$$

It has roots

$$\gamma = im \pm \sqrt{m^2 - (j-m)^2 - (l-m)^2}.$$

Stable (unstable respectively) modes come from $\mathrm{Re}(\gamma) < 0$ ($\mathrm{Re}(\gamma) > 0$ respectively). So for $(j, l) \in \mathbb{Z}^2$ with $(j-m)^2 + (l-m)^2 < m^2$, there are stable and unstable modes corresponding to roots $im \mp \sqrt{m^2 - (j-m)^2 - (l-m)^2}$ respectively. Similar computation shows that the auxiliary equation for (4.3.1) has only purely imaginary roots. So the above computation gives

Proposition 4.1. *Let $a = diag(im, -im)$ and $J = \pm e^{-(x+y)a}$, where $m > 0$ is an integer. Let*

$$B\mathbb{Z}_m = \{(j,l) \in \mathbb{Z}^2 \mid (j-m)^2 + (l-m)^2 < m^2\}.$$

Then:

(1) *The unstable subspace of the linearization of the Ward equation at J is*

$$\bigoplus \{W_{jl}^+ \mid (j,l) \in B\mathbb{Z}_m\},$$

where W_{jl}^+ is spanned by

$$\eta_{jl}^+(c) = e^{\sqrt{m^2-(j-m)^2-(l-m)^2}\ t} \begin{pmatrix} 0 & ce^{i(jx+ly+mt)} \\ -\bar{c}e^{-i(jx+ly+mt)} & 0 \end{pmatrix}$$

with constant $c \in \mathbb{C}$.

(2) *The stable subspace at J is*

$$\bigoplus \{W_{jl}^- \mid (j,l) \in B\mathbb{Z}_m\},$$

where W_{jl}^- is spanned by

$$\eta_{jl}^-(c) = e^{-\sqrt{m^2-(j-m)^2-(l-m)^2}\ t} \begin{pmatrix} 0 & ce^{i(jx+ly+mt)} \\ -\bar{c}e^{-i(jx+ly+mt)} & 0 \end{pmatrix}$$

with $c \in \mathbb{C}$.

Let $z = re^{i\theta} \in \mathbb{C}\backslash\mathbb{R}$, $f(w)$ a meromorphic function on $\mathbb{C}$, $q(w) = \begin{pmatrix} 1 \\ f(w) \end{pmatrix}$, $w = x + zu + z^{-1}v$, and $\pi(x,y,t)$ the Hermitian projection of $\mathbb{C}^2$ onto $\mathbb{C}q(w)$. Let ψ be the extended solution given by (4.1) and $J_0 = \psi^{-1}|_{\lambda=0} = e^{-a(x+y)}$ the associated Ward map. Consider the Bäcklund transformation $h_{z,\pi} * \psi$. We will find conditions on z and $f(w)$ so that $h_{z,\pi} * J_0$ is doubly periodic in space variables. By Theorem 2.2, we have

$$\psi_1 = h_{z,\pi} * \psi = h_{z,\tilde{\pi}}\psi, \tag{4.4}$$

where $\mathrm{Im}\tilde{\pi} = \mathbb{C}\tilde{q}$ and

$$\begin{aligned} \tilde{q}(x,y,t) &= \psi(z)q(w) \\ &= e^{((1-z)x+(1+z-z^2)u-v)a} \begin{pmatrix} 1 \\ f(w) \end{pmatrix} \\ &\sim \begin{pmatrix} 1 \\ e^{2im((z-1)x+(z^2-z-1)u+v)} f(w) \end{pmatrix}. \end{aligned}$$

Here "$q_1 \sim q_2$" means $\mathbb{C}q_1 = \mathbb{C}q_2$. From the formula

$$J_1 = h_{z,\pi} * J_0 = J_0 \frac{1}{|z|}(\bar{z}\tilde{\pi} + z\tilde{\pi}^{\perp}),$$

we see that it is doubly periodic if and only if $\tilde{q}$ is. For this purpose, we try the following form of

$$f(w) = e^{2im(\alpha - z)w},$$

where $\alpha \in \mathbb{C}$ is a constant. Substitute this into $\tilde{q}(x, y, t)$ to get

$$\begin{aligned} e^{2im((z-1)x+(z^2-z-1)u+v)} f(w) &= e^{2im((\alpha-1)x+((\alpha-1)z-1)u+\alpha z^{-1}v)} \\ &= e^{im(2(\alpha-1)x+((\alpha-1)z-\alpha z^{-1}-1)y+((\alpha-1)z+\alpha z^{-1}-1)t)}. \end{aligned}$$

It is doubly periodic in x and y with period 2π if and only if

$$\begin{cases} 2m(\alpha - 1) := -j \in \mathbb{Z}, \\ m((\alpha-1)z - \alpha z^{-1} - 1) := -l \in \mathbb{Z}. \end{cases} \tag{4.5}$$

From (4.5.1), we have $\alpha = \frac{2m-j}{2m}$. Compute the imaginary part of (4.5.2) to get

$$(\alpha - 1) r \sin\theta + \alpha r^{-1} \sin\theta = 0.$$

Since $r > 0$, we see $0 < \alpha < 1$. This implies that $0 < j < 2m$, and $r = \sqrt{\frac{2m-j}{j}}$. By (4.5.2) again, we have

$$m((\alpha-1)z - \alpha z^{-1} - 1) = -\sqrt{j(2m-j)}\cos\theta - m = -l.$$

It follows that

$$\sqrt{j(2m-j)}\cos\theta = l - m.$$

Hence l must satisfy

$$|l - m| < \sqrt{j(2m-j)}, \tag{4.6}$$

and $\cos\theta = \frac{l-m}{\sqrt{j(2m-j)}}$. It is easy to verify that the conditions for (j, l) are equivalent to $(j, l) \in \mathbb{Z}^2$, $(j-m)^2 + (l-m)^2 < m^2$, i.e., $(j, l) \in B\mathbb{Z}_m$. Therefore if we choose the following data: $(j, l) \in B\mathbb{Z}_m$, $z = re^{i\theta}$ with $r = \sqrt{\frac{2m-j}{j}}$, $\cos\theta = \frac{l-m}{\sqrt{j(2m-j)}}$, $\sin\theta > 0$, $\alpha = \frac{2m-j}{2m}$, then $\mathrm{Im}\tilde{\pi}(x, y, t) = \mathbb{C}\tilde{q}(x, y, t)$, where

$$\tilde{q}(x, y, t) = \begin{pmatrix} 1 \\ e^{\sqrt{m^2-(j-m)^2-(l-m)^2}\ t}\ e^{-i(jx+ly+mt)} \end{pmatrix}$$

is doubly periodic in x and y. It follows that

$$\tilde{\pi}(x, y, t) = \frac{1}{1+e^{2A}} \begin{pmatrix} 1 & e^{A} e^{i(jx+ly+mt)} \\ e^{A} e^{-i(jx+ly+mt)} & e^{2A} \end{pmatrix}.$$

where $A = \sqrt{m^2 - (j-m)^2 - (l-m)^2}\ t$. Therefore we obtain the following Ward map from $T^2 \times \mathbb{R}$ to $SU(2)$:

$$\begin{aligned} J_1 &= e^{-(x+y)a} \frac{1}{|z|} (\bar{z}\tilde{\pi}(x, y, t) + z\tilde{\pi}^{\perp}(x, y, t)) \\ &= e^{-(x+y)a} (e^{-i\theta}\tilde{\pi}(x, y, t) + e^{i\theta}\tilde{\pi}^{\perp}(x, y, t)). \end{aligned} \tag{4.7}$$

We now analyze the asymptotic behavior of J_1 as $t \to \pm\infty$. It is easy to see that

$$\tilde{\pi} \to \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{as } t \to +\infty,$$

and

$$\tilde{\pi} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{as } t \to -\infty.$$

So

$$J_1 \to e^{-(x+y)a} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad \text{as } t \to +\infty,$$

and

$$J_1 \to e^{-(x+y)a} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \quad \text{as } t \to -\infty.$$

Thus $J_1 : T^2 \times \mathbb{R} \to SU(2)$ is a heteroclinic Ward map. To construct homoclinic maps, we apply Bäcklund transformation again.

Choose $z_2 = -\bar{z}$, and $\pi_2(x,y,t)$ Hermitian projection of $\mathbb{C}^2$ onto $\mathbb{C}q_2$, where $q_2 = \begin{pmatrix} 1 \\ f_2(w_2) \end{pmatrix}$, $f_2(w_2) = e^{2im(\alpha+\bar{z})w_2}$, $\alpha = \frac{2m-j}{2m}$, and $w_2 = x - \bar{z}u - \bar{z}^{-1}v$. Now apply Bäcklund transformation to ψ_1 (defined by (4.4)) generated by $h_{-\bar{z},\pi_2(x,y,t)}$ to get

$$\psi_2 = h_{-\bar{z},\pi_2(x,y,t)} * \psi_1 = h_{-\bar{z},\tilde{\pi}_2(x,y,t)}\psi_1,$$

where $\mathrm{Im}\tilde{\pi}_2(x,y,t) = \mathbb{C}\tilde{q}_2(x,y,t)$ and

$$\begin{aligned} \tilde{q}_2 &= \psi_1(-\bar{z})q_2 = h_{z,\tilde{\pi}(x,y,t)}(-\bar{z})\psi(-\bar{z}) \begin{pmatrix} 1 \\ f_2(w_2) \end{pmatrix} \\ &\sim \left(\mathrm{I} + \frac{\bar{z}-z}{-2\bar{z}}\tilde{\pi}(x,y,t) \right) \begin{pmatrix} 1 \\ e^{A}e^{-i(jx+(2m-l)y+mt)} \end{pmatrix}. \end{aligned} \tag{4.8}$$

Here $A = \sqrt{m^2-(j-m)^2-(l-m)^2}\ t$. Note that $\tilde{q}(x,y,t)$ is doubly periodic in x and y. Therefore

$$\begin{aligned} J_2 = &\ J_0 \frac{1}{|z|}(\bar{z}\tilde{\pi} + z\tilde{\pi}^{\perp}) \frac{1}{|z_2|}(\bar{z}_2\tilde{\pi}_2 + z_2\tilde{\pi}_2^{\perp}) \\ = &\ J_0(e^{-i\theta}\tilde{\pi} + e^{i\theta}\tilde{\pi}^{\perp})(-e^{i\theta}\tilde{\pi}_2 - e^{-i\theta}\tilde{\pi}_2^{\perp}) \end{aligned}$$

is a Ward map from $T^2 \times \mathbb{R}$ to $SU(2)$, where $J_0 = e^{-(x+y)a}$.

Next we study the asymptotic behavior of J_2. First look at the behavior of J_2 as $t \to -\infty$. Set

$$\xi = e^{\sqrt{m^2-(j-m)^2-(l-m)^2}\ t}, \quad f_1 = e^{-i(jx+ly+mt)}, \quad f_2 = e^{-i(jx+(2m-l)y+mt)}.$$

Then $\lim_{t\to-\infty} \xi = 0$. Write

$$\tilde{q}_1 = \tilde{q} = \begin{pmatrix} 1 \\ \xi f_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \xi f_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So the projection $\tilde{\pi}_1 = \tilde{\pi}$ onto $\mathbb{C}\tilde{q}_1$ is

$$\tilde{\pi}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \xi \begin{pmatrix} 0 & \bar{f}_1 \\ f_1 & 0 \end{pmatrix} + O(\xi^2).$$

Write $\alpha_1 = \frac{z-\bar{z}}{2\bar{z}}$. Then by (4.8) we have

$$\begin{aligned}
\tilde{q}_2 &= (\mathrm{I} + \alpha_1 \tilde{\pi}_1) \begin{pmatrix} 1 \\ \xi f_2 \end{pmatrix} \\
&= (\mathrm{I} + \alpha_1 \tilde{\pi}_1) \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \xi f_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} 1 + \alpha_1 \\ 0 \end{pmatrix} + \alpha_1 \xi f_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \xi f_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\xi^2) \\
&\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_1 \xi f_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \beta_2 \xi f_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\xi^2),
\end{aligned}$$

where $\beta_1 = \frac{\alpha_1}{1+\alpha_1}$, $\beta_2 = \frac{1}{1+\alpha_1}$. So the projection $\tilde{\pi}_2$ onto $\mathbb{C}\tilde{q}_2$ is

$$\tilde{\pi}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \xi \begin{pmatrix} 0 & \bar{\beta}_1 \bar{f}_1 \\ \beta_1 f_1 & 0 \end{pmatrix} + \xi \begin{pmatrix} 0 & \bar{\beta}_2 \bar{f}_2 \\ \beta_2 f_2 & 0 \end{pmatrix} + O(\xi^2).$$

From the above computation, we see

$$\lim_{t \to -\infty} \tilde{\pi}_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, 2.$$

Substitute $\tilde{\pi}_i$ into J_2 to get

$$\begin{aligned}
J_2 &= J_0 (e^{-i\theta} \tilde{\pi} + e^{i\theta} \tilde{\pi}^{\perp})(-e^{i\theta} \tilde{\pi}_2 - e^{-i\theta} \tilde{\pi}_2^{\perp}) \\
&= J_0 \left(\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} + \xi \begin{pmatrix} 0 & -2i \sin\theta \ \bar{f}_1 \\ -2i \sin\theta \ f_1 & 0 \end{pmatrix} + O(\xi^2) \right) \\
&\quad \times \left(\begin{pmatrix} -e^{i\theta} & 0 \\ 0 & -e^{-i\theta} \end{pmatrix} + \xi \begin{pmatrix} 0 & -2i \sin\theta \ \bar{\beta}_1 \bar{f}_1 \\ -2i \sin\theta \ \beta_1 f_1 & 0 \end{pmatrix} \right. \\
&\qquad \left. + \xi \begin{pmatrix} 0 & -2i \sin\theta \ \bar{\beta}_2 \bar{f}_2 \\ -2i \sin\theta \ \beta_2 f_2 & 0 \end{pmatrix} + O(\xi^2) \right) \\
&= J_0 \left(-I + \xi \begin{pmatrix} 0 & c_1 \bar{f}_1 \\ -\bar{c}_1 f_1 & 0 \end{pmatrix} + \xi \begin{pmatrix} 0 & c_2 \bar{f}_2 \\ -\bar{c}_2 f_2 & 0 \end{pmatrix} + O(\xi^2) \right),
\end{aligned}$$

where $c_1, c_2 \in \mathbb{C}$ are constants. It follows that $\lim_{t \to -\infty} J_2 = -J_0$. Note that

$$\xi \begin{pmatrix} 0 & c_i \bar{f}_i \\ -\bar{c}_i f_i & 0 \end{pmatrix}, \quad i = 1, 2$$

is equal to the unstable mode $\eta^{+}_{j_i, l_i}(c_i)$ at $-J_0$ given in Proposition 4.1, where $(j_1, l_1) = (j, l)$ and $(j_2, l_2) = (j, 2m - l)$. In other words, we have shown

$$\lim_{t \to -\infty} \left(J_2 + J_0 + J_0 \sum_{i=1}^{2} \eta^{+}_{j_i, l_i}(c_i) \right) = 0. \tag{4.9}$$

To analyze the asymptotic behavior of J_2 as $t \to +\infty$, we set

$$\rho = e^{-\sqrt{m^2-(j-m)^2-(l-m)^2}\ t},\ h_1 = e^{i(jx+ly+mt)},\ h_2 = e^{i(jx+(2m-l)y+mt)}.$$

Then $\lim_{t\to+\infty} \rho = 0$. A similar computation implies that

(1) $\tilde{q}_1$ is parallel to $\begin{pmatrix} \rho h_1 \\ 1 \end{pmatrix}$.

(2)

$$\tilde{\pi}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \rho \begin{pmatrix} 0 & h_1 \\ \bar{h}_1 & 0 \end{pmatrix} + O(\rho^2),$$

and

$$\tilde{\pi}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \rho \begin{pmatrix} 0 & \gamma_1 h_1 \\ \bar{\gamma}_1 \bar{h}_1 & 0 \end{pmatrix} + \rho \begin{pmatrix} 0 & \gamma_2 h_2 \\ \bar{\gamma}_2 \bar{h}_2 & 0 \end{pmatrix} + O(\rho^2)$$

for some constants γ_1, γ_2.

(3)

$$J_2 = J_0 \left(-I + \rho \begin{pmatrix} 0 & d_1 h_1 \\ -\bar{d}_1 \bar{h}_1 & 0 \end{pmatrix} + \rho \begin{pmatrix} 0 & d_2 h_2 \\ -\bar{d}_2 \bar{h}_2 & 0 \end{pmatrix} + O(\rho^2) \right)$$

for some constants d_1, d_2. It follows that $\lim_{t\to+\infty} J_2 = -J_0$.

(4)

$$\lim_{t\to+\infty} \left(J_2 + J_0 + J_0 \sum_{i=1}^{2} \eta^-_{j_i,l_i}(d_i) \right) = 0. \tag{4.10}$$

Formulas (4.9) and (4.10) imply that $J_2 : T^2 \times \mathbb{R} \to SU(2)$ is a homoclinic Ward map.

Applying Bäcklund transformations even times, with pairs of poles and Hermitian projections chosen as above, we obtain more homoclinic Ward maps. The case for $m < 0$ is similar. We summarize the above discussion to give:

Theorem 4.2. *Let m be a nonzero integer, $a = diag(im, -im)$, and $J_0 = e^{-(x+y)a}$. Choose $(j_{2k-1}, l_{2k-1}) \in \mathbb{Z}^2$ such that*

$$(j_{2k-1} - m)^2 + (l_{2k-1} - m)^2 < m^2, \quad m < l_{2k-1}$$

and $(j_{2k-1}, l_{2k-1}) \neq (j_{2h-1}, l_{2h-1})$ for $1 \le k < h \le N$. Let $(j_{2k}, l_{2k}) = (j_{2k-1}, 2m - l_{2k-1})$, $z_s = \sqrt{\frac{2m-j_s}{j_s}}\ e^{i\theta_s}$ with $\cos\theta_s = \operatorname{sgn}(m)\frac{(l_s-m)}{\sqrt{j_s(2m-j_s)}}$, and $\sin\theta_s = \operatorname{sgn}(m)\sqrt{1-\cos^2\theta}$, $s = 1, \ldots, 2N$. Let π_s be the Hermitian projection onto $\mathbb{C}\begin{pmatrix} 1 \\ f_s(w_s) \end{pmatrix}$, where $f_s(w_s) = e^{i(2m-j_s-2mz_s)w_s}$ and $w_s = x + z_s u + z_s^{-1} v$. Let

$$J_{2N}(x,y,t) = h_{z_{2N},\pi_{2N}} * (\cdots * (h_{z_1,\pi_1} * J_0(x,y,t)) \cdots) \tag{4.11}$$

be the Ward map obtained by applying $2N$ Bäcklund transformations to J_0. Then $J_{2N} : T^2 \times \mathbb{R} \to SU(2)$ is a homoclinic Ward map. Moreover, $J_{2N} \to (-1)^N J_0$ as $t \to \pm\infty$.

Proof. We have shown the $N = 1$ case. For general N, we use induction and the calculation is similar. $\square$

The above construction can be generalized to $SU(n)$ model easily:

Corollary 4.3. *Let m, p be integers, $m \neq 0$, $1 \le p \le n-1$,*

$$a = \begin{pmatrix} i(n-p)mI_p & 0 \\ 0 & -ipmI_{n-p} \end{pmatrix} \in su(n),$$

$\psi = e^{((1-\lambda)x+(1+\lambda-\lambda^2)u-v)a}$ the extended solution, and $J_0 = e^{-(x+y)a}$ the associated Ward map. Choose $(j_{2k-1}, l_{2k-1}) \in \mathbb{Z}^2$ such that

$$\left(j_{2k-1} - \frac{nm}{2}\right)^2 + \left(l_{2k-1} - \frac{nm}{2}\right)^2 < \left(\frac{nm}{2}\right)^2, \quad \frac{nm}{2} < l_{2k-1}$$

and $(j_{2k-1}, l_{2k-1}) \neq (j_{2h-1}, l_{2h-1})$ for $1 \le k < h \le N$. Let $(j_{2k}, l_{2k}) = (j_{2k-1}, nm - l_{2k-1})$, $z_s = \sqrt{\frac{nm-j_s}{j_s}} e^{i\theta_s}$ with $\cos\theta_s = \mathrm{sgn}(m)\frac{(l_s - nm/2)}{\sqrt{j_s(nm-j_s)}}$, and $\sin\theta_s = \mathrm{sgn}(m)\sqrt{1-\cos^2\theta}$, $s = 1, \ldots, 2N$. Let $\pi_s(x, y, t)$ be the Hermitian projection of $\mathbb{C}^n$ onto

$$\mathbb{C}(1, \cdots, 1, f_s(w_s), \cdots f_s(w_s))^T,$$

where 1 is repeated p-times, $f_s(w_s) = e^{i(nm-j_s-nmz_s)w_s}$ is repeated $(n-p)$-times, $w_s = x + z_s u + z_s^{-1} v$. Let

$$J_{2N}(x, y, t) = h_{z_{2N},\pi_{2N}} * (\cdots * (h_{z_1,\pi_1} * J_0(x, y, t)) \cdots) \tag{4.12}$$

be the Ward map obtained by applying $2N$ Bäcklund transformations to J_0. Then $J_{2N} : T^2 \times \mathbb{R} \to SU(n)$ is a homoclinic Ward map. Moreover, $J_{2N} \to (-1)^N J_0$ as $t \to \pm\infty$.

The method discussed above can also produce Ward maps, which are homoclinic to (time) periodic orbits. There are only some minor changes in the construction, so we just list the main steps for the $SU(2)$ model.

- Let $m > 0$ be an integer, and $b = \mathrm{diag}(im, -im)$. Then
$$\psi = e^{(x+(\lambda+2)u)b}$$
is an extended solution, and the associated Ward map is
$$J_0 = \psi^{-1}|_{\lambda=0} = e^{-(x+2u)b} = e^{-(x+y+t)b},$$
which is triply periodic in the variables x, y, t.
- Set $\eta = J^{-1}\delta J$. Then the linearization of the Ward equation at J_0 is
$$\eta_{tt} - \eta_{xx} - \eta_{yy} + [b, \eta_x + 2\eta_y - 2\eta_t] = 0.$$
The unstable subspace of the linearization of the Ward equation at J_0 is $\bigoplus W_{jl}^+$, where $(j,l) \in \mathbb{Z}^2$, $(j-m)^2 + (l-2m)^2 < m^2$, and W_{jl}^+ is spanned by
$$e^{\sqrt{m^2-(j-m)^2-(l-2m)^2}\ t} \begin{pmatrix} 0 & ce^{i(jx+ly+2mt)} \\ -\bar{c}e^{-i(jx+ly+2mt)} & 0 \end{pmatrix}$$

with constant $c \in \mathbb{C}$. The stable subspace at J_0 is $\bigoplus W_{jl}^-$, where (j,l) satisfies the same condition, and W_{jl}^- is spanned by

$$e^{-\sqrt{m^2-(j-m)^2-(l-2m)^2}\ t}\begin{pmatrix} 0 & ce^{i(jx+ly+2mt)} \\ -\bar{c}e^{-i(jx+ly+2mt)} & 0 \end{pmatrix}$$

with constant $c \in \mathbb{C}$.

- Choose $(j,l) \in \mathbb{Z}^2$ with $(j-m)^2+(l-2m)^2 < m^2$. Apply Bäcklund transformation $h_{z_1,\pi_1} * \psi$, where $z_1 = re^{i\theta}$ with $r = \sqrt{\frac{2m-j}{j}}$, $\cos\theta = \frac{l-2m}{\sqrt{j(2m-j)}}$, $\sin\theta > 0$, $\pi_1(x,y,t)$ is the Hermitian projection of $\mathbb{C}^2$ onto $\mathbb{C}\begin{pmatrix} 1 \\ e^{2im\alpha_1 w_1}\end{pmatrix}$, $\alpha_1 = \frac{2m-j}{2m}$, and $w_1 = x + z_1 u + z_1^{-1} v$. Then
$$\psi_1 = h_{z_1,\pi_1} * \psi = h_{z_1,\tilde{\pi}_1}\psi,$$
where $\tilde{\pi}_1$ is the projection onto
$$\psi(z_1)\mathrm{Im}\pi_1 = \mathbb{C}\begin{pmatrix} 1 \\ e^{\sqrt{m^2-(j-m)^2-(l-2m)^2}\,t} e^{-i(jx+ly+2mt)}\end{pmatrix}.$$
- Choose $(j_2, l_2) = (j, 4m-l) \in \mathbb{Z}^2$. Apply Bäcklund transformation again to get
$$\psi_2 = h_{z_2,\pi_2} * \psi_1 = h_{z_2,\tilde{\pi}_2}\psi_1.$$
Here $z_2 = -\bar{z}_1$, and $\pi_2(x,y,t)$ is the Hermitian projection onto $\mathbb{C}q$, where $q = \begin{pmatrix} 1 \\ e^{2im\alpha_2 w_2}\end{pmatrix}$, $\alpha_2 = \alpha_1$, and $w_2 = x - \bar{z}_1 u - \bar{z}_1^{-1} v$. Then $\tilde{\pi}_2(x,y,t)$ is the projection onto
$$\mathbb{C}h_{z_1,\tilde{\pi}_1}(-\bar{z}_1)\begin{pmatrix} 1 \\ e^{\sqrt{m^2-(j-m)^2-(l-2m)^2}\,t} e^{-i(jx+(4m-l)y+2mt)}\end{pmatrix}.$$
- $$J_2 = \psi_2^{-1}|_{\lambda=0} = J_0(e^{-i\theta}\tilde{\pi}_1 + e^{i\theta}\tilde{\pi}_1^{\perp})(-e^{i\theta}\tilde{\pi}_2 - e^{-i\theta}\tilde{\pi}_2^{\perp})$$
is a Ward map from $T^2\times\mathbb{R}$ to $SU(2)$. Analyzing the asymptotic behavior of J_2 as $t \to \pm\infty$, we see that J_2 is transversal and homoclinic to the periodic orbit $-J_0$. Applying Bäcklund transformations even times with pairs of poles and Hermitian projections chosen similarly, we obtain more Ward maps which are homoclinic to $\pm J_0$.

The construction of homoclinic orbits to (time) periodic solutions for the $SU(n)$ model is similar. Thus we have

Theorem 4.4. *Let m, p be integers, $m \neq 0$, $1 \le p \le n-1$,*
$$b = \begin{pmatrix} i(n-p)mI_p & 0 \\ 0 & -ipmI_{n-p}\end{pmatrix} \in su(n),$$

$\psi = e^{(x+(\lambda+2)u)b}$ the extended solution, and $J_0 = e^{-(x+y+t)b}$ the associated Ward map. Choose $(j_{2k-1}, l_{2k-1}) \in \mathbb{Z}^2$ such that

$$\left(j_{2k-1} - \frac{nm}{2}\right)^2 + (l_{2k-1} - nm)^2 < \left(\frac{nm}{2}\right)^2, \quad nm < l_{2k-1}$$

and $(j_{2k-1}, l_{2k-1}) \neq (j_{2h-1}, l_{2h-1})$ for $1 \leq k < h \leq N$. Let $(j_{2k}, l_{2k}) = (j_{2k-1}, 2nm - l_{2k-1})$, $z_s = \sqrt{\frac{nm-j_s}{j_s}} e^{i\theta_s}$ with $\cos\theta_s = \mathrm{sgn}(m)\frac{(l_s-nm)}{\sqrt{j_s(nm-j_s)}}$, and $\sin\theta_s = \mathrm{sgn}(m)\sqrt{1-\cos^2\theta}$, $s = 1, \ldots, 2N$. Let $\pi_s(x,y,t)$ be the Hermitian projection of $\mathbb{C}^n$ onto

$$\mathbb{C}(1, \cdots, 1, f_s(w_s), \cdots f_s(w_s))^T,$$

where 1 is repeated p-times, $f_s(w_s) = e^{i(nm-j_s)w_s}$ is repeated $(n-p)$-times, and $w_s = x + z_s u + z_s^{-1} v$. Let

$$J_{2N}(x,y,t) = h_{z_{2N},\pi_{2N}} * (\cdots * (h_{z_1,\pi_1} * J_0(x,y,t)) \cdots)$$

be the Ward map obtained by applying $2N$ Bäcklund transformations to J_0. Then $J_{2N} : T^2 \times \mathbb{R} \to SU(n)$ is a homoclinic Ward map. Moreover, $J_{2N} \to (-1)^N J_0$ as $t \to \pm\infty$.

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